#4.2 Direct Proof and Counterexample – II
Rational Numbers

* Definition: A real number r is rational iff $\exists$ integers a and b such that $r = \frac{a}{b}$ and $b \neq 0$.

* Answer the following questions with justification.

1. Is $10/3$ a rational number?

Ans. Yes. As $\frac{10}{3} = \frac{a}{b}$ with $a = 10, b = 3 \in \mathbb{Z}$ and $b = 3 \neq 0$.

2. Is $-\frac{5}{39}$ a rational number?

Ans. Yes. As $-\frac{5}{39} = \frac{-5}{39} = \frac{a}{b}$ with $a = -5, b = 39 \in \mathbb{Z}$ and $b = 39 \neq 0$.

3. Is 0.281 a rational number?

Ans. Yes. As $0.281 = \frac{281}{1000} = \frac{a}{b}$ with $a = 281$, ~~an~~ $b = 1000 \in \mathbb{Z}$ and $b = 1000 \neq 0$.

4. Is 7 a rational number?

Ans. Yes. As $7 = \frac{7}{1} = \frac{a}{b}$ with $a = 7, b = 1 \in \mathbb{Z}$ and $b = 1 \neq 0$.

5. Is 0 a rational number?

Ans Yes. As $0 = \frac{0}{1} = \frac{a}{b}$ with $a = 0, b = 1 \in \mathbb{Z}$ and $b = 1 \neq 0$.

6. Is $2/0$ a rational number?

Ans. No. As $2/0 = a/b$ with $a = 2, b = 0 \in \mathbb{Z}$ but "$b = 0$".

7. Is $2/0$ an irrational number?

Ans. No. As $2/0$ is not defined (ie) is not a number.

8. Is $0.121212\ldots\ldots$ a rational number?

Ans. Yes.

Let $x = 0.121212\ldots\ldots$

$\therefore 100x = 12.121212\ldots\ldots$

$\therefore 100x - x = \{12.121212\ldots - 0.121212\ldots\ldots = 12$

$99x = 12 \Rightarrow x = \frac{12}{99} \Rightarrow 0.121212\ldots = \frac{12}{99}$

9. If m and n are integers and neither m nor n is zero, is $(m+n)/mn$ a rational number?

Ans. Yes. As $\frac{m+n}{mn} = \frac{a}{b}$ with $a = m+n$, $b = mn \in \mathbb{Z}$

Since sum and product of integers is integer.
Also $b = mn \neq 0$ since both m and n are non-zero.

* Zero Product Property:
If neither of two real numbers is zero, then their product is also not zero.

* Prove that every integer is a rational number.

Solution: To prove, $\forall x \in \mathbb{R}$, if $x \in \mathbb{Z}$ then $x \in \mathbb{Q}$.

Proof: Suppose $x \in \mathbb{R}$ such that $x \in \mathbb{Z}$.

Then $x = \frac{x}{1} = \frac{a}{b}$ with $a = x$, $b = 1 \in \mathbb{Z}$ and $b = 1 \neq 0$.

$\therefore x \in \mathbb{Q}$.

Hence proved.

* Prove that the sum of any two rational numbers is rational.

Solution: To prove, $\forall x, y \in \mathbb{R}$, if $x, y \in \mathbb{Q}$ then $x+y \in \mathbb{Q}$.

Proof: Suppose $x, y \in \mathbb{R}$ such that $x, y \in \mathbb{Q}$.

By definition of rational numbers,

$x = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ and $b \neq 0$

and $y = \frac{c}{d}$ for some $c, d \in \mathbb{Z}$ and $d \neq 0$.

$$\therefore x+y = \frac{a}{b} + \frac{c}{d}$$

$$= \frac{ad+bc}{bd}$$

$$= \frac{p}{q}$$

where $p = ad+bc$, $q = bd \in \mathbb{Z}$ since sum and product of integers is integer.

Also $q = bd \neq 0$ by zero product property.

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* Prove that the double of a rational number is rational.

Solution: To prove, $\forall x \in \mathbb{R}$, if $x \in \mathbb{Q}$ then $2x \in \mathbb{Q}$. (3)

Proof 1: Suppose $x \in \mathbb{R}$ such that $x \in \mathbb{Q}$.

By definition of rational numbers,

$x = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ and $b \neq 0$.

$\therefore 2x = \frac{2a}{b} = \frac{p}{q}$

where $p = 2a$, $q = b \in \mathbb{Z}$ and $q = b \neq 0$.

$\therefore 2x$ is rational.

Hence proved

Proof 2: Suppose $x \in \mathbb{R}$ such that $x \in \mathbb{Q}$

$2x = x + x$

and sum of rational numbers is rational.

$\therefore 2x \in \mathbb{Q}$.

Hence proved.

Exercise - 4.2

* ~~Answer the following question~~

* Write each number as a ratio of two integers.

1. $\frac{35}{6} = \frac{-35}{6}$

2. $4.6037 = \frac{46037}{10000}$

3. $\frac{4}{5} + \frac{2}{9} = \frac{36+10}{45} = \frac{46}{45}$

4. $0.373737\ldots\ldots$

Let $x = 0.373737\ldots$

$\therefore 100x = 37.3737\ldots\ldots$

$\therefore 100x - x = 37.373737\ldots\ldots - 0.373737\ldots\ldots$

$99x = 37$

$\therefore x = \frac{37}{99}$

$\therefore 0.373737\ldots\ldots = \frac{37}{99}$

5. $0.565656.....$

Let $x = 0.565656......$

$\therefore 100x = 56.565656.....$

$\therefore 100x - x = 56.565656..... - 0.565656.....$

$99x = 56$

$\therefore x = \frac{56}{99}$

$\therefore 0.565656.... = \frac{56}{99}$

6. $320.5492492492.......$

Let $x = 320.5492492492......$

$\therefore 10x = 3205.492492492......$

and $10000x = 3205492.492492492......$

$\therefore 10000x - 10x = 3205492.492492..... - 3205.492492.....$

$9990x = 3202287$

$\therefore x = \frac{3202287}{9990}$

$\therefore 320.5492492492.... = \frac{3202287}{9990}$

7. $52.467216721672.........$

Let $x = 52.467216721672.......$

$\therefore 10000x = 524672.16721672......$

and $100000000x = 5246721672.16721672......$

$\therefore 100000000x - 10000x = 5246721672.16721672......$

$- 524672.16721672......$

$99990000x = 5246197000$

$\therefore x = \frac{5246197000}{99990000}$

$\therefore 52.467216721672..... = \frac{5246197}{99990}$

* Prove that square of any rational number is rational.

Solution: To prove, $\forall x \in \mathbb{R}$, if $x \in \mathbb{Q}$ then $x^2 \in \mathbb{Q}$.

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Proof: Suppose $x \in \mathbb{R}$ such that $x \in \mathbb{Q}$.

By definition of rational numbers,

$x = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ and $b \neq 0$.

$\therefore x^2 = \frac{a^2}{b^2} = \frac{p}{q}$

where $p = a^2, q = b^2 \in \mathbb{Z}$ since product of two integers is integer. and $q = b^2 \neq 0$ by zero product property.

$\therefore x^2 \in \mathbb{Q}$.

Hence proved.

* Prove that the negative of any rational number is rational.

Solution: To prove, $\forall x \in \mathbb{R}$, if $x \in \mathbb{Q}$ then $-x \in \mathbb{Q}$.

Proof: Suppose $x \in \mathbb{R}$ such that $x \in \mathbb{Q}$

By definition of rational numbers,

$x = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ and $b \neq 0$.

$\therefore -x = -\frac{a}{b} = \frac{p}{q}$

where $p = -a$, $q = b \in \mathbb{Z}$ and $q = b \neq 0$.

$\therefore -x \in \mathbb{Q}$.

Hence proved.

* Prove or disprove, the product of any two rational numbers is a rational number.

Solution: The given statement is true.

To prove, $\forall x, y \in \mathbb{R}$, if $x, y \in \mathbb{Q}$ then $xy \in \mathbb{Q}$.

Proof: Suppose $x, y \in \mathbb{R}$ such that $x, y \in \mathbb{Q}$.

By definition of rational numbers,

$x = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ and $b \neq 0$

and $y = \frac{c}{d}$ for some $c, d \in \mathbb{Z}$ and $d \neq 0$.

$\therefore xy = \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd} = \frac{p}{q}$

where $p = ac$; $q = bd \in \mathbb{Z}$ since product of integers is an integer.

and $q = bd \neq 0$ by zero product property. ⑥

$\therefore xy \in \mathbb{Q}$

Hence proved.

→ Prove or disprove, the quotient of any two rational numbers is rational.

Solution: The given statement is false.

Counterexample:

Let $x = 1$ and $y = 0$ $\Rightarrow \frac{x}{y} = \frac{1}{0}$ ← not defined.

Here $x, y \in \mathbb{Q}$ but $xy \notin \mathbb{Q}$.

→ Prove or disprove, the difference of any two rational number is is a rational number.

Solution: The given statement is true.

To prove, $\forall x, y \in \mathbb{R}$, if $x, y \in \mathbb{Q}$ then $x - y \in \mathbb{Q}$.

Proof: Suppose $x, y \in \mathbb{R}$ such that $x, y \in \mathbb{Q}$.

By definition of rational numbers,

$x = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ and $b \neq 0$

and $y = \frac{c}{d}$ for some $c, d \in \mathbb{Z}$ and $d \neq 0$.

$$\therefore x - y = \frac{a}{b} - \frac{c}{d}$$

$$= \frac{ad - bc}{bd}$$

$$= \frac{p}{q}$$

where $p = ad - bc$; $q = bd \in \mathbb{Z}$ since product and difference of integers is Integers.

and $q = bd \neq 0$ by zero product property.

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18: Prove or disprove; if r and s are any two rational numbers, then $\frac{r+s}{2}$ is rational.

Solution: The given statement is true. 7

To prove, $\forall r, s \in \mathbb{R}$; if $r, s \in \mathbb{Q}$ then $\frac{r+s}{2} \in \mathbb{Q}$.

Proof: ~~~~ Suppose $r, s \in \mathbb{R}$ such that $r, s \in \mathbb{Q}$.

By definition of rational nos,

$r = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ and $b \neq 0$

and $s = \frac{c}{d}$ for some $c, d \in \mathbb{Z}$ and $d \neq 0$.

$$\therefore \frac{r+s}{2} = \frac{\frac{a}{b} + \frac{c}{d}}{2} = \frac{ad+bc}{2bd} = \frac{p}{q}$$

where $p = ad+bc$, $q = 2bd \in \mathbb{Z}$ since product and sum of intergers is an integer;

and $q = 2bd \neq 0$ by zero product property.

#4.3 Direct Proof and Counterexample II: Divisibility

* Notation:

$d \mid n$ is read as "d divides n".

* Definition: If n and d are integers and $d \neq 0$,
$d \mid n$ iff $\exists$ an integer k such that $n = dk$.

Instead of "d divides n", we can also say that
n is a multiple of d or
d is a factor of n or
d is a divisor of n or
d divides n is divisible by d.

* Answer the following questions with justification:

1. Is 21 divisible by 3?

Ans. Yes. As $21 = 3 \times 7$.

2. Does 5 divide 40?

Ans. Yes. As $40 = 5 \times 8$.

3. Does $7 \mid 42$?

Ans. Yes. As $42 = 7 \times 6$.

4. Is 32 a multiple of -16?

Ans. Yes. As $32 = (-16) \times (-2)$.

5. Is 6 a factor of 54?

Ans. Yes. As $54 = 6 \times 9$.

6. Is 7 a factor of -7?

Ans. Yes. As $-7 = 7 \times (-1)$.

7. If k is any nonzero integer, does k divide 0?

Ans. Yes. As $0 = k \cdot 0$.

* Prime Numbers and Divisibility:

An alternative way to define a prime number is to say that an integer $n \geq 1$ is prime iff its only positive integer divisors are 1 and itself.

* Transitivity of Divisibility:

Prove that for all integers a, b and c, if $a|b$ and $b|c$ then $a|c$.

Proof: Suppose a, b and $c \in \mathbb{Z}$ such that $a|b$ and $b|c$.

By definition of divisibility,

$b = ak_1$ for some $k_1 \in \mathbb{Z}$.

and $c = bk_2$ for some $k_2 \in \mathbb{Z}$.

Put $b = ak_1$ in $c = bk_2$ we get

$c = (ak_1)k_2$

$c = a(k_1k_2)$

$c = ak$ where $k = k_1k_2 \in \mathbb{Z}$ since product of integers is an integer

$\therefore a|c$

Hence proved.

* Prove or disprove, for all integers a and b, if $a|b$ and $b|a$ then $a=b$.

Solution: The given statement is false.

Counterexample: Let $a=2$ and $b=-2$

Then $2|-2$ as $-2 = 2 \times (-1)$ and

$-2|2$ as $2 = (-2) \times (-1)$ but $2 \neq -2$.

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Exercise - 4.3

~~1. Is 52 div~~

* Answer each of the following with justification.

1. Is 52 divisible by 13?

Ans. Yes. As $52 = 13 \times 4$.

2. Does $7 | 56$?

Ans. Yes. As $56 = 7 \times 8$.

3. Does $5 | 0$?

Ans. Yes. As $0 = 5 \times 0$.

4. Does 3 divide $(3k+1)(3k+2)(3k+3)$?

Ans. Yes. As $(3k+1)(3k+2)(3k+3) = 3(3k+1)(3k+2)(k+1)$
$= 3k'$

where $k' = (3k+1)(3k+2)(k+1) \in \mathbb{Z}$.
Since sum and product of integers is ~~again~~ an integer.

5. Is $6m(2m+10)$ divisible by 4?

Ans. Yes. As $6m(2m+10) = 2 \times 6m(m+5)$
$= 4 \times 3m(m+5)$
$= 4k$

where $k = 3m(m+5) \in \mathbb{Z}$
Since sum and product of integers is an integer.

6. Is 29 a multiple of 3?

Ans. No. As $\frac{29}{3} = 9.66\ldots\ldots \notin \mathbb{Z}$.

7. Is -3 a factor of 66?

Ans. Yes. As $66 = (-3) \times (-22)$.

8. Is $6a(a+b)$ a multiple of $3a$?

Ans. Yes. As $6a(a+b) = 3a \times 2(a+b)$
$= 3a \times k$

where $k = 2(a+b) \in \mathbb{Z}$ since sum and product of integers is an integer.

9. Is 4 a factor of $2a \cdot 34b$?

Ans. Yes. As $2a \cdot 34b = 4 \cdot 17ab$

10. Does $7 \mid 34$?

Ans. ~~Yes. As~~ No. As $\frac{34}{7} = \cancel{4.08556}\ 4.857\ldots\ldots \notin \mathbb{Z}$.

11. Does $13 \mid 73$?

Ans. No. As $\frac{73}{13} = 5.615\ldots\ldots \notin \mathbb{Z}$

12. If $n = 4k+1$, does 8 divide n^2-1?

Ans. Yes. As $n^2-1 = (4k+1)^2-1$

$$= 16k^2+8k+1-1$$
$$= 16k^2+8k$$
$$= 8(2k^2+k)$$
$$= 8k'$$

where $k' = 2k^2+k \in \mathbb{Z}$ since sum and product of integers is an integer.

13. If $n = 4k+3$, does 8 divide n^2-1?

Ans. Yes. As $n^2-1 = (4k+3)^2-1$

$$= 16k^2+24k+9-1$$
$$= 8(2k^2+3k+1)$$
$$= 8k'$$

where $k' = 2k^2+3k+1 \in \mathbb{Z}$ since sum and product of integers is an integer.

14. Prove that for all integers a and b, if $a \mid b$ then $a \mid (-b)$.

Proof: Suppose $a, b \in \mathbb{Z}$ such that $a \mid b$.

By definition of divisibility,

$b = ak$ for some $k \in \mathbb{Z}$.

$\therefore -b = -(ak)$

$$= a(-k)$$
$$= ak'$$

where $k' = -k \in \mathbb{Z}$ since negative of an integer is an integer.

15. ~~For a~~ Prove that, for all integers a, b and c if $a|b$ and $a|c$ then $a|(b+c)$. and $a|(b-c)$. (12)

Proof! Suppose $a, b \in \mathbb{Z}$ such that $a|b$ and $a|c$

By definition of divisibility,

$b = ak_1$ for some $k_1 \in \mathbb{Z}$

$c = ak_2$ for some $k_2 \in \mathbb{Z}$.

$\therefore b+c = ak_1 + ak_2$

$= a(k_1+k_2)$

$= ak$ where $k = k_1+k_2 \in \mathbb{Z}$ since sum of integers is an integer.

$\therefore a|b+c$.

Also, $b-c = ak_1 - ak_2$

$= a(k_1-k_2)$

$= ak$

where $k = k_1-k_2 \in \mathbb{Z}$ since difference of Integers is an integer.

$\therefore a|b-c$

Hence proved.

17. Prove or disprove, the negative of any multiple of 3 is a multiple of 3.

Solution: The given statement is true.

To prove, $\forall n \in \mathbb{Z}$, if n is multiple of 3 then $-n$ is multiple of 3.

Proof! ~~Let~~ Suppose $n \in \mathbb{Z}$ ~~st~~ such that n is multiple of 3.

By definition of divisibility,

$n = 3k$ for some $k \in \mathbb{Z}$.

$\therefore -n = -3k$

$= 3(-k)$

$= 3k'$ where $k' = -k \in \mathbb{Z}$ since negative of any integers is an integer.

$\therefore -n$ is multiple of 3.

Hence proved.

18. Prove or disprove, for all integers a and b, if $3 \mid a+b$ then $3 \mid a-b$.

Solution: The given statement is false.

Counterexample: Let $a=2$ and $b=1$

Then $a+b = 2+1 = 3$ and $a-b = 2-1 = 1$

$\therefore$ $3 \mid a+b$ but $3 \nmid a-b$.

19. Prove that, for all integers a, b and c, if a divides b then a divides bc.

Proof: Suppose $a, b, c \in \mathbb{Z}$ such that $a \mid b$.

By definition of divisibility,

$b = ak$ for some $k \in \mathbb{Z}$.

$\therefore bc = (ak)c$

$= a(kc)$

$= ak'$ where $k' = kc \in \mathbb{Z}$ since product of integers is an integer.

$\therefore a \mid bc$

Hence proved.

20. Prove that, the sum of any three consecutive integers is divisible by 3.

Solution: Suppose $n, (n+1), (n+2)$ be any three consecutive integers.

$\therefore n + (n+1) + (n+2) = 3n+3$

$= 3(n+1)$

$= 3k$

where $k = n+1 \in \mathbb{Z}$

$\therefore$ Hence proved.

21. Prove that, the product of any two even integer is a multiple of 4.

Solution: To prove, $\forall m, n \in \mathbb{Z}$, if m and n are even then mn is a multiple of 4.

Proof: ~~be~~ Suppose $m, n \in \mathbb{Z}$ such that m and n are even.

By definition of even integers,

$m = 2k_1$ for some $k_1 \in \mathbb{Z}$

and $n = 2k_2$ for some $k_2 \in \mathbb{Z}$.

$\therefore mn = (2k_1)(2k_2) = 4k_1k_2 = 4k$ where $k = k_1k_2 \in \mathbb{Z}$ since......

Hence proved.